Gauss

1.2 – Gaussian Elimination

#7 Solve the system by Gaussian elimination. – use an augmented matrix

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ \boxed{2} & 1 & -2 & -2 & -2 \\ \boxed{-1} & \boxed{2} & -4 & 1 & 1 \\ \boxed{3} & \boxed{0} & 0 & -3 & -3 \end{array}\right] \quad ① \quad ②$$

$R_2 \to R_2 + 2R_3$

$$\begin{array}{ccccc} 2 & 1 & -2 & -2 & -2 \\ -2 & 4 & -8 & 2 & 2 \\ \hline 0 & 5 & -10 & 0 & 0 \end{array}$$

$R_3 \to R_3 + R_1$

$$\begin{array}{ccccc} -1 & 2 & -4 & 1 & 1 \\ 1 & -1 & 2 & -1 & -1 \\ \hline 0 & 1 & -2 & 0 & 0 \end{array}$$

$R_4 \to R_4 + 3R_3$

$$\begin{array}{ccccc} 3 & 0 & 0 & -3 & -3 \\ -3 & 6 & -12 & 3 & 3 \\ \hline 0 & 6 & -12 & 0 & 0 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 5 & -10 & 0 & 0 \\ 0 & \boxed{1} & -2 & 0 & 0 \\ 0 & \boxed{6} & -12 & 0 & 0 \end{array}\right] \quad ②$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 5 & -10 & 0 & 0 \\ 0 & 6 & -12 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - 5R_2$

$$\begin{array}{ccccc} 0 & 5 & -10 & 0 & 0 \\ 0 & -5 & 10 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}$$

$R_4 \to R_4 - 6R_2$

$$\begin{array}{ccccc} 0 & 6 & -12 & 0 & 0 \\ 0 & -6 & 12 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\to x - \underline{y} + 2z - w = -1$

$\to \underline{y - 2z = 0}$

← Row echelon form via Gaussian elimination

$y = 2z \Rightarrow \underline{x} - w = -1$

$x = w - 1$ Let $s = w$, $t = z$

$y = 2z$

Then

$$\boxed{\begin{aligned} x &= s-1 \\ y &= 2t \\ z &= t \\ w &= s \end{aligned}} \Rightarrow \vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} s-1 \\ 2t \\ t \\ s \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ 2t \\ t \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

rref

<u>**Definitions:**</u> **Row echelon form** (#1-3 below) and **reduced row echelon form** (#1-4 below) of a matrix – to be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. This is a **leading 1.**
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

$$\left[\begin{array}{cccc|c} \underline{1} & -1 & 2 & -1 & -1 \\ 0 & \underline{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\begin{array}{ccccc} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & -1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \to x - w = -1 \Rightarrow x = w - 1 \\ \to y - 2z = 0 \Rightarrow y = 2z \\ \\ \\ \end{array}$$

reduced row echelon form

via Gauss Jordan elimination

We started with and now have

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right] \qquad \left[\begin{array}{cccc|c} \underline{1} & 0 & 0 & -1 & -1 \\ 0 & \underline{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

positions that correspond to leading 1s in rref are <u>pivots</u>

Leading 1s correspond to leading variables

x, y

z, w are free variables

columns that contain leading 1s are pivot columns

Definitions: **Leading variables** correspond to the leading 1's in the augmented matrix for a linear system. **Free variables** are the remaining variables in the linear system.

Every variable is either leading or free.

Definition: In a **homogeneous linear system**, the constant terms are all zero. That is, it has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

Theorem 1.2.1 Free Variable Theorem for Homogeneous Systems
If a homogeneous linear system has $\overset{4}{n}$ unknowns, and if the reduced row echelon form of its augmented matrix has $\underset{3}{r}$ nonzero rows, then the system has $n - r$ free variables.

$4-3=1$

Theorem 1.2.2 A homogeneous linear system with more unknowns than equations has infinitely many solutions.

Pf: The # of equations is the # of rows in the augmented matrix for the system. the # of leading 1s is no more than the # of rows. The # of columns in the coefficient matrix is more than the # of rows. Thus, the system has at least one free variable and so has infinitely many solutions.

#28 What condition, if any, must a, b, and c satisfy for the linear system to be consistent? A solution exists.

$$x+3y+z=a$$
$$-x-2y+z=b$$
$$3x+7y-z=c$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ \boxed{-1} & -2 & 1 & b \\ \boxed{3} & \boxed{7} & -1 & c \end{array}\right]$$

$R_2 \Rightarrow R_2 + R_1$

$$\begin{array}{cccc} -1 & -2 & 1 & b \\ 1 & 3 & 1 & a \\ \hline 0 & 1 & 2 & a+b \end{array}$$

$R_3 \Rightarrow R_3 - 3R_1$

$$\begin{array}{cccc} 3 & 7 & -1 & c \\ -3 & -9 & -3 & -3a \\ \hline 0 & -2 & -4 & c-3a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ \boxed{0} & 1 & 2 & a+b \\ \boxed{0} & \boxed{-2} & -4 & c-3a \end{array}\right]$$

$R_3 \Rightarrow R_3 + 2R_2$

$$\begin{array}{cccc} 0 & -2 & -4 & c-3a \\ 0 & 2 & 4 & 2a+2b \\ \hline 0 & 0 & 0 & -a+2b+c \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & -a+2b+c \end{array}\right]$$

$R_3: 0 = -a+2b+c$

The system is consistent if $-a+2b+c=0$

OR if $a=2b+c$.

$R_2 \Rightarrow R_2 + \frac{1}{2}R_1$

creates fractions

we could:

$R_2 \Rightarrow R_1 + 2R_2$

okay but not elementary

Solve the given linear system by any method.

#16

$$2x - y - 3z = 0$$
$$-x + 2y - 3z = 0$$
$$x + y + 4z = 0$$

$$\rightarrow \begin{array}{cccc} & \textcircled{4} & \textcircled{3} & \\ \left[\begin{array}{c} 2 \\ -1 \\ 1 \end{array}\right. & \begin{array}{c} -1 \\ 2 \\ 1 \end{array} & \begin{array}{c} -3 \\ -3 \\ 4 \end{array} & \left|\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] \\ \textcircled{1} & \textcircled{2} & & \end{array}$$

writing the zeros is optional

$R_3 \Longleftrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & -3 \\ 2 & -1 & -3 \end{bmatrix}$$

$R_2 \Rightarrow R_2 + R_1$

$$\begin{array}{ccc} -1 & 2 & -3 \\ 1 & 1 & 4 \\ \hline 0 & 3 & 1 \end{array}$$

$R_3 \Rightarrow R_3 + 2R_2$

$$\begin{array}{ccc} 2 & -1 & -3 \\ -2 & 4 & -6 \\ \hline 0 & 3 & -9 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 3 & -9 \end{bmatrix}$$

$R_3 \Rightarrow R_3 - R_2$

$$\begin{array}{ccc} 0 & 3 & -9 \\ 0 & -3 & -1 \\ \hline 0 & 0 & -10 \end{array}$$

Then $R_3 \Rightarrow -\frac{1}{10} R_3$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_3$

$$\begin{array}{ccc} 0 & 3 & 1 \\ 0 & 0 & -1 \\ \hline 0 & 3 & 0 \end{array}$$

Then $R_2 \Rightarrow \frac{1}{3} R_2$

$R_1 \Rightarrow R_1 - 4R_3$

$$\begin{array}{ccc} 1 & 1 & 4 \\ 0 & 0 & -4 \\ \hline 1 & 1 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \Rightarrow x = 0 \\ \Rightarrow y = 0 \\ \Rightarrow z = 0 \end{array}$$

$$\boxed{(x, y, z) = (0, 0, 0)}$$

This is called the <u>trivial</u> solution

#19

$2x + 2y + 4z = 0$ use $x+y+2z=0$

$w - y - 3z = 0$

$2w + 3x + y + z = 0$

$-2w + x + 3y - 2z = 0$

Solutions look like $(5,-5,5,0)$

$$\begin{array}{cccc} 0 & 1 & 1 & 2 \\ 1 & 0 & -1 & -3 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \\ ① & ② & ③ & \end{array} \qquad R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \end{bmatrix}$$

$R_3 \to R_3 + R_4$

$$\begin{array}{cccc} 2 & 3 & 1 & 1 \\ -2 & 1 & 3 & -2 \\ \hline 0 & 4 & 4 & -1 \end{array}$$

$R_4 \to R_4 + 2R_1$

$$\begin{array}{cccc} -2 & 1 & 3 & -2 \\ 2 & 0 & -2 & -6 \\ \hline 0 & 1 & 1 & -8 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 4 & -1 \\ 0 & 1 & 1 & -8 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$

$$\begin{array}{cccc} 0 & 4 & 4 & -1 \\ 0 & -4 & -4 & -8 \\ \hline 0 & 0 & 0 & -9 \end{array}$$

$R_4 \to R_4 - R_2$

$$\begin{array}{cccc} 0 & 1 & 1 & -8 \\ 0 & -1 & -1 & -2 \\ \hline 0 & 0 & 0 & -10 \end{array}$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

Formal steps: $R_3 \to -\frac{1}{9}R_3$, $R_4 \to -\frac{1}{10}R_4$, then $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form

Formal steps:

$R_2 \to R_2 - 2R_3$

$R_1 \to R_1 + 3R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = x_3 \leftarrow t = x_3$

$x_2 = -x_3$

$x_4 = 0$

$(x_1, x_2, x_3, x_4) = (t, -t, t, 0)$

#**23** The augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

a. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$

This has a
unique solution

b. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right]$

Consistent.
Not a unique solution

c. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow$ $0 = 1$ is a contradiction.
this has no solution

d. $\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{array}\right]$

$\leftarrow \begin{cases} \text{If } * = 0, \\ \text{no information} \\ \text{If } * \neq 0,\ z = 0 \end{cases}$ soln

$\leftarrow z = *$

is $* \neq 0$
contradiction
if $* = 0$

d) Not enough information